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Generalized Invariant Monotonicity and Invexity of Non-differentiable Functions

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Abstract. This paper is devoted to the study of relationships between several kinds of generalized invexity of locally Lipschitz functions and generalized monotonicity of corresponding Clarke's subdifferentials. In particular, some necessary and sufficient conditions of being a locally Lipschitz function invex, quasiinvex or pseudoinvex are given in terms of momotonicity, quasimonotonicity and pseudomonotonicity of its Clarke's subdifferential, respectively. As an application of our results, the existence of the solutions of the variational-like inequality problems as well as the mathematical programming problems (MP) is given. Our results extend and unify the well known earlier works of many authors.

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1. Introduction

It is well known that the generalized monotonicity of set-valued mapping plays an important role in studying the existence and the sensitivity analysis of solutions for variational inequalities, variational inclusions, and complementarity problems. Convexity also plays a central role in mathematical economics, engineering, management sciences and optimization. In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson (1981). His initial result inspired a great deal of subsequent work which has greatly expanded the role and applications of invexity in non-linear optimization and other branches of pure and applied sciences. In fact he has shown that the Kuhn–Tuker conditions are sufficient for optimality of non-linear programming problems under invexity conditions. Kaul and Kaur (1985) presented the notions of strictly pseudoinvex, and quasinvex functions, and investigated their applications in non-linear programming. Weir and Mond (1988) and Weir and Jeyakumar (1988) have studied the basic properties of preinvex functions and their applications in optimization, Pini (1991) introduced the concepts of prepseudoinvex and prequasiinvex functions and established the relationships between invexity and generalized invexity. Mohan and Neogy (1995) showed that under certain assumptions, an invex function is preinvex and a quasiinvex function is prequasiinvex. More recently, characterizations and applications of preinvex functions, semistrictly preinvex functions, prequasiinvex functions, and semistrictly prequasiinvex functions were studied by Yang and Li (2001) and Yang et al. (2001). The relationships between generalized convexity of functions and generalized monotone operators have been investigated by many authors; see Karamardian and Schaible (1990), Correa et al. (1992), Penot and Quang (1992), Luc (1993a,b, 1994), Penot and Sach (1997), Fan et al. (2003), Bianchi and Schaible (2004) and the recent Hadjisavvas et al.'s Handbook (2005). Similar to the case of convexity, Garzon et al. (2003), Yang et al. (2003) and Lue and Xu (2004) obtained analogous results for invexity and generalized invexity. Motivated by the work of Yang et al. (2003, 2005) and Fan et al. (2003) we study relationships between several kinds of generalized invexity of locally Lipschitz functions and generalized monotonicity of corresponding Clarke's subdifferentials. Indeed by using the techniques of Yang et al. (2001, 2003, 2005), some necessary and sufficient conditions of being a locally Lipschitz function invex, quasiinvex or pseudoinvex are given in terms of momotonicity, quasimonotonicity and pseudomonotonicity of its Clarke's subdifferential, respectively. As an application of our results, the existence of the solutions of the variational-like inequality problems as well as the mathematical programming problems (MP) is given.

2. Preliminaries

Let X be a real Banach space endowed with a norm $\|\,.\,\|$ and X^* its dual space with a norm $\|\,.\,\|_*$. We denote by 2^{X^*}, $\langle .,. \rangle$, $[x, y]$, and (x, y) the family of all non-empty subsets of X^*, the dual pair between X and X^*, the line segment for $x, y \in X$, and the interior of $[x, y]$, respectively. Let K be a non-empty open subset of X, $T: X \to 2^{X^*}$ a set-valued mapping, $\eta: X \times X \to X$ a vector-valued function, and $f: X \to \mathbb{R}$ a non-differentiable real-valued function. The following concepts and results are taken from Clarke et al. (1998).

DEFINITION 2.1. Let f be locally Lipschitz at a given point $x \in X$ and v any vector in X. The Clarke's generalized directional derivative of f at x in the direction v, denoted by $f^{\circ}(x; v)$, is defined by

$$f^{\circ}(x; v) = \limsup_{y \to x, t \downarrow 0} \frac{f(y+tv) - f(y)}{t}.$$

DEFINITION 2.2. Let f be locally Lipschitz at a given point $x \in X$ and v any vector in X. The Clarke's generalized subdifferential of f at x, denoted by $\partial^c f(x)$, is defined as follows:

$$\partial^c f(x) = \{\xi \in X^*: f^{\circ}(x; v) \geqslant \langle \xi, v \rangle, \forall v \in X\}.$$

LEMMA 2.1. *Let f be locally Lipschitz with a constant L at $x \in X$. Then:*

(1) *$\partial^c f(x)$ is a non-empty convex, weak*–compact subset of X^* and $\| \xi \|_* \leqslant L$ for each $\xi \in \partial^c f(x)$.*
(2) *For every $v \in X$, $f^{\circ}(x; v) = \max\{\langle \xi, v \rangle : \xi \in \partial^c f(x)\}$.*
(3) *$\xi \in \partial^c f(x)$ if and only if $f^{\circ}(x; v) \geqslant \langle \xi, v \rangle$ $\forall v \in X$.*
(4) *If $\{x_i\}, \{\xi_i\}$ are sequences in X and X^* such that $\xi_i \in \partial^c f(x_i)$ for each i, and if x_i converges to x, and ξ is a weak*–cluster point of the sequence $\{\xi_i\}$, then we have $\xi \in \partial^c f(x)$.*

LEMMA 2.2 (Mean Value Theorem). *Let x and y be point in X and suppose that f is Lipschitz near each points of a non-empty closed convex set containing the line segment $[x, y]$. Then there exists a point $u \in (x, y)$ such that*

$$f(x) - f(y) \in \langle \partial^c f(u), x - y \rangle.$$

3. Invexity and Invariant Monotonicity

In this section, we establish the relationships between (strict, strong) invexity of f and (strict, strong) invariant monotonicity of its Clarke's generalized subdifferential mapping $\partial^c f$.

DEFINITION 3.1. A subset K of X is said to be invex with respect to η: $X \times X \to X$ if, for any $x, y \in K$ and $\lambda \in [0, 1]$, $y + \lambda \eta(x, y) \in K$.

In this paper, we suppose that X is a Banach space, $\eta: X \times X \to X$ is a vector-valued function, $K \subset X$ is an invex set with respect to η, $f : X \to \mathbb{R}$ is a function, and $T : X \to 2^{X^*}$ is a set-valued mapping.

Following Definitions 2.3, 2.6 of Yang et al. (2003) and 3.2 of Fan et al. (2003), we present the following definition.

DEFINITION 3.2. Let $T: X \to 2^{X^*}$ be a set-valued mapping:

(1) T is said to be invariant monotone (IM) on K with respect to η if for any $x, y \in K$ and any $u \in T(x), v \in T(y)$, one has

$$\langle v, \eta(x, y)\rangle + \langle u, \eta(y, x)\rangle \leqslant 0;$$

(2) T is said to be strictly invariant monotone (SIM) on K with respect to η if for any $x, y \in K$ with $x \neq y$ and any $u \in T(x), v \in T(y)$, one has

$$\langle v, \eta(x, y)\rangle + \langle u, \eta(y, x)\rangle < 0;$$

(3) T is said to be strongly invariant monotone (SGIM) on K with respect to η if there exist a constant $\alpha > 0$ such that for any $x, y \in K$ and any $u \in T(x), v \in T(y)$, one has

$$\langle v, \eta(x, y)\rangle + \langle u, \eta(y, x)\rangle \leqslant -\alpha(\| \eta(x, y) \|^2 + \| \eta(y, x) \|^2).$$

REMARK 3.1. Every strictly invariant monotone map is an invariant monotone map with respect to η, but the converse is not necessarily true, The following example due to Yang et al. (2003) shows that strictly invariant monotone map differs from strongly invariant map in general.

EXAMPLE 3.1. Define the maps F and η as

$$F(x) = (1 + \cos x_1, 1 + \cos x_2), \quad x = (x_1, x_2) \in K,$$
$$\eta(x, y) = [(\sin x_1 - \sin y_1)/\cos y_1, (\sin x_2 - \sin y_2)/\cos y_2], \quad x, y \in K,$$

where $K = (0, \pi/2) \times (0, \pi/2)$. Then F is a strictly invariant monotone map with respect to η, but F is not strongly invariant monotone map with respect to η on K. If $K' = (\pi/6, \pi/3) \times (\pi/6, \pi/3)$ then F is a strongly invariant monotone map with respect to η on K'.

By Definition 2.4 of Yang et al. (2003) and Definition 3.1 of Fan et al. (2003), we present the following definition.

DEFINITION 3.3. Let K be an invex set with respect to η and $f: K \to \mathbb{R}$:

(1) f is said to be preinvex (PX) with respect to η on K if for any $x, y \in K$ and $\lambda \in [0, 1]$, one has

$$f(y + \lambda\eta(x, y)) \leqslant \lambda f(x) + (1 - \lambda) f(y);$$

(2) f is said to be strictly preinvex (SPX) with respect to η on K if for any $x, y \in K$ with $x \neq y$ and for any $\lambda \in (0, 1)$, one has

$$f(y+\lambda\eta(x, y)) < \lambda f(x) + (1-\lambda) f(y);$$

(3) f is said to be strongly preinvex (SGPX) with respect to η on K if there exist a constant $\alpha > 0$ such that for any $x, y \in K$ and $\lambda \in [0, 1]$, one has

$$f(y+\lambda\eta(x, y)) \leqslant \lambda f(x) + (1-\lambda) f(y) - \alpha\lambda(1-\lambda) \| \eta(x, y) \|^2 .$$

The following assumptions are useful in the sequel.

ASSUMPTION A. *Let K be an invex set with respect to η, and let $f: K \to \mathbb{R}$. Then,*

$$f(y+\eta(x, y)) \leqslant f(x) \quad \text{for any } x, y \in K.$$

ASSUMPTION C. *Let $\eta: X \times X \to X$. Then, for any $x, y \in X$ and for any $\lambda \in [0, 1]$.*

$$\eta(y, y+\lambda\eta(x, y)) = -\lambda\eta(x, y),$$
$$\eta(x, y+\lambda\eta(x, y)) = (1-\lambda)\eta(x, y).$$

REMARK 3.2. Recently Yang et al. (2005) have shown that if $\eta: X \times X \to X$ satisfies Assumption C, then

$$\eta(y+\lambda\eta(x, y), y) = \lambda\eta(x, y).$$

By definition of invex function in Banach spaces and Definition 3.1 of Fan et al. (2003), we give the following definition.

DEFINITION 3.4. Let f be locally Lipschitz on K, Then:

(1) function $f: K \to \mathbb{R}$ is said to be invex (IX) with respect to η on K if for any $x, y \in K$ and any $\zeta \in \partial^c f(y)$, one has

$$\langle \zeta, \eta(x, y) \rangle \leqslant f(x) - f(y);$$

(2) function $f: K \to \mathbb{R}$ is said to be strictly invex (SIX) with respect to η on K if for any $x, y \in K$ with $x \neq y$ and any $\zeta \in \partial^c f(y)$, one has

$$\langle \zeta, \eta(x, y) \rangle < f(x) - f(y);$$

(3) f is said to be strongly invex (SGIX) with respect to, η on K if there exists a constant $\alpha > 0$ such that for any $x, y \in K$ and any $\zeta \in \partial^c f(y)$, one has

$$\langle \zeta, \eta(x, y) \rangle + \alpha \parallel \eta(x, y) \parallel^2 \leqslant f(x) - f(y).$$

Here, we show that different kinds of invexity with respect to η imply monotonicity of subdifferential with respect to η.

LEMMA 3.1. *Let f be locally Lipschitz on K. If f is (strongly, strictly) invex with respect to η on K, then $\partial^c f$ is (strongly, strictly) invariant monotone with respect to η on K.*

Proof. We prove only the assertion strongly and with $\alpha = 0$ and by replacing $\leqslant$ and $\geqslant$ by $<$ and $>$, the other cases can be proved similarly. Suppose that f is strongly invex with respect to η on K. Then for $x, y \in K$, $\zeta \in \partial^c f(y)$ and $\gamma \in \partial^c f(x)$, by strong invexity of f, we have

$$f(x) - f(y) \geqslant \langle \zeta, \eta(x, y) \rangle + \alpha \parallel \eta(x, y) \parallel^2,$$

and

$$f(y) - f(x) \geqslant \langle \gamma, \eta(y, x) \rangle + \alpha \parallel \eta(y, x) \parallel^2.$$

By adding these two relations, we have

$$0 \geqslant \langle \zeta, \eta(x, y) \rangle + \langle \gamma, \eta(y, x) \rangle + \alpha(\parallel \eta(y, x) \parallel^2 + \parallel \eta(y, x) \parallel^2).$$

Then, $\partial^c f$ is strongly invariant monotone with respect to η on K. □

In the following theorem we will show the inverse implications of the above Lemma hold in the presence of Assumptions A and C.

THEOREM 3.1. *Let f be locally Lipschitz on K and f and η satisfy Assumptions A and C. If $\partial^c f$ is (strongly, strictly) invariant monotone with respect to η on K, then f is (strongly, strictly) invex with respect to η on K.*

Proof. We prove only the assertion strongly and with $\alpha = 0$ and by replacing $\leqslant$ and $\geqslant$ by $<$ and $>$, the other cases can be proved similarly. Let $\partial^c f$ be strongly invariant monotone with respect to η on K with constant $\alpha > 0$, $x, y \in K$ and $w \in \partial^c f(y)$. Let $z = y + \frac{1}{2}\eta(x, y)$, then by the Mean Value Theorem, there exist λ_1, λ_2 such that $0 < \lambda_2 < \frac{1}{2} < \lambda_1 < 1$ and there exist $\zeta \in \partial^c f(u)$ and $\gamma \in \partial^c f(v)$ such that

$$f(y + \eta(x, y)) - f(z) = \langle \zeta, y + \eta(x, y) - z \rangle = \frac{1}{2} \langle \zeta, \eta(x, y) \rangle \tag{3.1}$$

and

$$f(z) - f(y) = \langle \gamma, z - y \rangle = \frac{1}{2} \langle \gamma, \eta(x, y) \rangle, \tag{3.2}$$

where $u = y + \lambda_1 \eta(x, y)$ and $v = y + \lambda_2 \eta(x, y)$. Since $\partial^c f$ is a strongly invariant monotone operator and $\zeta \in \partial^c f(u)$ and $w \in \partial^c f(y)$, we have

$$\langle \zeta, \eta(y, u) \rangle + \langle w, \eta(u, y) \rangle \leqslant -\alpha(\| \eta(y, u) \|^2 + \| \eta(u, y) \|^2). \tag{3.3}$$

On the other hand by Assumption C and Remark 3.2, we have

$$\eta(u, y) = \eta(y + \lambda_1 \eta(x, y), y) = \lambda_1 \eta(x, y),$$

and

$$\eta(y, u) = \eta(y, y + \lambda_1 \eta(x, y)) = -\lambda_1 \eta(x, y).$$

If we replace these values in (3.3) and divide by λ_1, we obtain

$$\langle \zeta, \eta(x, y) \rangle \geqslant 2\alpha\lambda_1 \| \eta(x, y) \|^2 + \langle w, \eta(x, y) \rangle. \tag{3.4}$$

In a similar way, we have

$$\langle \gamma, \eta(y, v) \rangle + \langle w, \eta(v, y) \rangle \leqslant -\alpha(\| \eta(y, v) \|^2 + \| \eta(v, y) \|^2), \tag{3.5}$$

where $\eta(v, y) = \lambda_2 \eta(x, y)$ and $\eta(y, v) = -\lambda_2 \eta(x, y)$. By replacing again these values in (3.5) and dividing by λ_2, we have

$$\langle \gamma, \eta(x, y) \rangle \geqslant 2\alpha\lambda_2 \| \eta(x, y) \|^2 + \langle w, \eta(x, y) \rangle. \tag{3.6}$$

Then by (3.4), (3.6), (3.1) and (3.2) we derive

$$f(y + \eta(x, y)) - f(z) \geqslant \alpha\lambda_1 \| \eta(x, y) \|^2 + \frac{1}{2} \langle w, \eta(x, y) \rangle,$$

and

$$f(z) - f(y) \geqslant \alpha\lambda_2 \| \eta(x, y) \|^2 + \frac{1}{2} \langle w, \eta(x, y) \rangle.$$

By adding these two relations, we obtain

$$\begin{aligned} f(y + \eta(x, y)) - f(y) &\geqslant \alpha(\lambda_1 + \lambda_2) \| \eta(x, y) \|^2 + \langle w, \eta(x, y) \rangle \\ &\geqslant \frac{\alpha}{2} \| \eta(x, y) \|^2 + \langle w, \eta(x, y) \rangle. \end{aligned}$$

But Assumption A implies that

$$f(x)-f(y)\geqslant\frac{\alpha}{2}\parallel\eta(x,y)\parallel^2+\langle w,\eta(x,y)\rangle.$$

Therefore, f is strongly invex. □

In the next part of this section we will establish the relationships between different kinds of invexity and preinvexity.
The following lemmas are similar to Theorem 2.1 of Mohan and Neogy (1995).

LEMMA 3.2. *Let f be locally Lipschitz on K and f and η satisfy Assumptions A and C. If f is (strongly, strictly) invex with respect to η on K, then f is (strongly, strictly) preinvex with respect to η on K.*

Proof. We prove only the assertion strongly and with $\alpha=0$ and by replacing $\leqslant$ and $\geqslant$ by $<$ and $>$, the other cases can be proved similarly. Suppose that f is strongly invex with respect to η on K with constant $\alpha>0, x, y\in K$ and $0<\lambda<1$ be given and set $\bar{x}=y+\lambda\eta(x,y)$. Note that $\bar{x}\in K$. By the strong invexity of f, we have

$$f(x)-f(\bar{x})\geqslant\langle\zeta,\eta(x,\bar{x})\rangle+\alpha\parallel\eta(x,\bar{x})\parallel^2\qquad\text{for each }\zeta\in\partial^c f(\bar{x}).\tag{3.7}$$

Similarly, the strong invexity condition applied to the pair $y,\bar{x}$ yields

$$f(y)-f(\bar{x})\geqslant\langle\zeta,\eta(y,\bar{x})\rangle+\alpha\parallel\eta(y,\bar{x})\parallel^2\qquad\text{for each }\zeta\in\partial^c f(\bar{x}).\tag{3.8}$$

We note that by Assumption C,

$$\eta(x,\bar{x})=(1-\lambda)\eta(x,y),\quad \eta(y,\bar{x})=-\lambda\eta(x,y).$$

Now, multiplying (3.7) by λ and (3.8) by $(1-\lambda)$ and adding, then

$$\begin{aligned}&\lambda f(x)+(1-\lambda)f(y)-f(\bar{x})\\&\quad\geqslant\langle\zeta,\lambda\eta(x,\bar{x})+(1-\lambda)\eta(y,\bar{x})\rangle+\alpha\lambda(1-\lambda)\parallel\eta(x,y)\parallel^2\\&\quad=\alpha\lambda(1-\lambda)\parallel\eta(x,y)\parallel^2.\end{aligned}$$

Hence, the conclusion follows. □

LEMMA 3.3. *Let η be continuous with respect to the second argument. Suppose that f is locally Lipschitz on K and f is (strongly) preinvex with respect to η, then f is (strongly) invex with respect to η on K.*

Proof. We prove only the assertion strongly and with $\alpha=0$, the other case can be proved similarly. Suppose that f is strongly preinvex with respect to η on K and let $x, y\in K$ and $\varepsilon>0$ be arbitrary. Let L be a Lipschitz constant of f in a neighborhood of y. Since η is continuous in the second argument, there exists $\delta > 0$ such that if $\| y - z \| \leqslant \delta$, then $|f(y) - f(z)| < \varepsilon/3$, $\| \eta(x, y) - \eta(x, z) \| < \varepsilon/3L$ and $(\| \eta(x, y) \|^2 - \| \eta(x, z) \|^2) < \varepsilon/3\alpha$, Hence for every $y \in K$, with $\| y - z \| < \gamma, 0 < \gamma < \delta$, and $0 < \lambda < \beta$ for a small enough $\beta > 0$, one has

$$\begin{aligned}\frac{f(z + \lambda\eta(x, y)) - f(z)}{\lambda} &\leqslant \frac{f(z + \lambda\eta(x, z)) - f(z)}{\lambda} + L \| \eta(x, y) - \eta(x, z) \| \\ &\leqslant \frac{\lambda f(x) + (1 - \lambda) f(z) - \alpha\lambda(1 - \lambda) \| \eta(x, z) \|^2 - f(z)}{\lambda} + \frac{\varepsilon}{3} \\ &\leqslant f(x) - f(y) - \alpha(1 - \lambda) \| \eta(x, y) \|^2 + \varepsilon,\end{aligned}$$

then

$$\begin{aligned}f^{\circ}(y, \eta(x, y)) = \inf_{\gamma > 0, \beta > 0} \sup_{\| z - y \| < \gamma, 0 < \lambda < \beta} \frac{f(z + \lambda\eta(x, y)) - f(z)}{\lambda} \\ \leqslant f(x) - f(Y) - \alpha \| \eta(x, y) \|^2 + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, hence for any $\zeta \in \partial^c f(y)$, we have

$$\langle \zeta, \eta(x, y) \rangle \leqslant f^{\circ}(y, \eta(x, y)) \leqslant f(x) - f(y) - \alpha \| \eta(x, y) \|^2.$$

Thus f is strongly preinvex. □

From Theorem 3.1 and Lemmas 3.1, 3.2, and 3.3 we deduce the following result.

THEOREM 3.2. *Let f be locally Lipschitz on K and f and η satisfy Assumptions A and C. If η is continuous with respect to the second argument, then the following assertions are equivalent:*

(1) *f is (strongly) invex on K with respect to η.*
(2) *$\partial^c f$ is (strongly) invariant monotone on K with respect to η.*
(3) *f is (strongly) preinvex on K with respect to η.*

4. Quasiinvexity and Invariant Quasimonotonicity

In this section, we establish the relationships between quasiinvexity of the non-differentiable function f and quasimonotonicity of its Clarke's subdifferential $\partial^c f$.

DEFINITION 4.1. A set-valued mapping T is said to be invariant quasimonotone (IQM) on K with respect to η if, for any $x, y \in K$ and any $u \in T(x), v \in T(y)$, one has

$$\langle u, \eta(y, x)\rangle > 0 \Rightarrow \langle v, \eta(x, y)\rangle \leqslant 0.$$

DEFINITION 4.2. f is said to be prequasiinvex (PQX) with respect to η on K if for any $x, y \in K, 0 \leqslant \lambda \leqslant 1$, one has

$$f(y + \lambda\eta(x, y)) \leqslant \max\{f(x), f(y)\}.$$

Trivially from Definitions 3.3 and 4.2, we have

$$(\text{SGPX}) \Rightarrow (\text{PX}) \Rightarrow (\text{PQX}).$$

DEFINITION 4.3. Let f be locally Lipschitz on K.Then f is said to be quasiinvex (QIX) with respect to η if for any $x, y \in K$ and any $\zeta \in \partial^c f(x)$, one has

$$f(y) \leqslant f(x) \Rightarrow \langle \zeta, \eta(y, x)\rangle \leqslant 0.$$

The following lemma is similar to Theorem 2.2 of Mohan and Neogy (1995) in non-differentiable setting.

LEMMA 4.1. *Let f be locally Lipschitz on K and η satisfy Assumption C. If, for ever pair of points $x, y \in K$, with $x \neq y$ and any $\zeta \in \partial^c f(x)$, one has*

$$(D) \quad f(y) < f(x) \Rightarrow \langle \zeta, \eta(y, x)\rangle \leqslant 0.$$

Then, the function f is prequasiinvex with respect to η on K. If η is continuous with respect to the second argument, the converse holds.

Proof. Suppose that condition (D) holds. Let $x, y \in K$, $f(x) \leqslant f(y)$ and consider the set

$$\Omega = \{z : z = y + \lambda\eta(x, y), \quad f(z) > f(y), \quad 0 \leqslant \lambda \leqslant 1\}.$$

In order to show that f is prequasiinvex, we have to show that $\Omega \neq \emptyset$. Note that if $\Omega \neq \emptyset$, then by continuity of f, the set

$$\Omega' = \{z : z = y + \lambda\eta(x, y), \quad f(z) > f(y), \quad 0 < \lambda < 1\}$$

is also non-empty. Hence, it suffices to show that $\Omega' = \emptyset$ to complete the proof.

Suppose now that $\bar{x} \in \Omega'$. Then we have $\bar{x} = y + \bar{\lambda}\eta(x, y)$, for some $0 < \bar{\lambda} < 1$ and $f(\bar{x}) > f(y) \geqslant f(x)$. Consider the pair $\bar{x}$ and x. By (D), for any $\zeta \in \partial^c f(\bar{x})$,

$$\langle \zeta, \eta(x, \bar{x}) \rangle \leqslant 0, \tag{4.1}$$

and for $\bar{x}, y$, we have

$$\langle \zeta, \eta(y, \bar{x}) \rangle \leqslant 0. \tag{4.2}$$

Hence by Assumption C and (4.1) and (4.2), we have

$$(1 - \bar{\lambda})\langle \zeta, \eta(x, y) \rangle \leqslant 0, \tag{4.3}$$

and

$$-\bar{\lambda}\langle \zeta, \eta(x, y) \rangle \leqslant 0. \tag{4.4}$$

Now (4.3) and (4.4), together with the fact that $0 < \bar{\lambda} < 1$, imply that for any $\zeta \in \partial^c f(\bar{x})$

$$\langle \zeta, \eta(x, y) \rangle = 0. \tag{4.5}$$

Note that (4.5) holds for any $\bar{x} \in \Omega'$. Now suppose that $\Omega' \neq \emptyset$, Let $\bar{x} \in \Omega'$ and let $\bar{x} = y + \bar{\lambda}\eta(x, y)$. By continuity of f , we can find $0 \leqslant \lambda^* < \bar{\lambda} < \hat{\lambda} < 1$ such that for all $\lambda \in (\lambda^*, \hat{\lambda})$, we have $f(y + \lambda\eta(x, y)) > f(y)$, therefore, $(y + \lambda\eta(x, y)) \in \Omega'$ and $f(y + \lambda^*\eta(x, y)) = f(y)$. Now, by the Mean Value Theorem there exist $\tilde{\lambda} \in (\lambda^*, \bar{\lambda})$ and $\gamma \in \partial^c f(y + \tilde{\lambda}\eta(x, y))$ such that

$$\begin{aligned} f(y + \bar{\lambda}\eta(x, y)) - f(y) &= f(y + \bar{\lambda}\eta(x, y)) - f(y + \lambda^*\eta(x, y)) \\ &= \langle \gamma, (\bar{\lambda} - \lambda^*)\eta(x, y) \rangle. \end{aligned}$$

The left-hand side is positive by our hypothesis, but the right-hand side is zero by (4.5). As $y + \tilde{\lambda}\eta(x, y) \in \Omega'$, hence, we have a contradiction, In the case where $f(y) \leqslant f(x)$ the proof is similar.

Conversely, suppose that f is prequasiinvex with respect to η on K and η is continuous with respect to the second argument. Let $x, y \in K$ and $f(y) < f(x)$, then by continuity of f there exists $\delta > 0$ such that for $\| z - x \| < \delta$, we have $f(y) < f(z)$. Then the prequasiinvexity of f implies that

$$f(z + \lambda\eta(y, z)) \leqslant f(z) \quad \text{for } \| z - x \| < \delta.$$

If L is a Lipschitz constant of f near the point x, then for $\delta > 0$ small enough, and for $\| z - x \| < \delta, 0 < \lambda < \delta$, one has

$$\frac{f(z+\lambda\eta(y,x)) - f(z)}{\lambda} \leqslant \frac{f(z+\lambda\eta(y,z)) - f(z)}{\lambda} + L \| \eta(y,z) - \eta(y,x) \|$$
$$\leqslant 0 + L \| \eta(y,z) - \eta(y,x) \| .$$

By continuity of η with respect to the second argument, we have

$$f^{\circ}(x, \eta(y,x)) = \lim_{\delta \to 0} \sup_{\|z-x\|<\delta, 0<\lambda<\delta} \frac{f(z+\lambda\eta(y,x)) - f(z)}{\lambda} \leqslant 0,$$

hence for any $\zeta \in \partial^c f(x)$, we obtain

$$\langle \zeta, \eta(y,x) \rangle \leqslant 0,$$

and condition (D) holds. □

From Lemma 4.1 and Definition 4.3, we can obtain the following corollary.

COROLLARY 4.1. *Let f be locally Lipschitz on K and η satisfy Assumption C. If f is quasiinvex with respect to η on K, then it is prequasiinvex with respect to η on K.*

In the following result, we obtain a refinement of Theorem 3.1 of Yang et al. (2003).

THEOREM 4.1. *Let f be locally Lipschitz on K. Suppose the following assertions hold:*

(1) *f is quasiinvex with respect to η on K.*
(2) *$\partial^c f$ is invariant quasimonotone with respect to η on K.*
(3) *f is prequasiinvex with respect to η on K.*

Then $(1) \Rightarrow (2)$. *If η satisfies Assumption C and, for all $x, y \in K$*

$$(B) \quad f(y + \eta(x,y)) \leqslant \max\{f(x), f(y)\},$$

then $(2) \Rightarrow (3)$.

Proof. Suppose (1) holds. Let $x, y \in K, u \in \partial^c f(x)$ and $v \in \partial^c f(y)$ be such that

$$\langle u, \eta(y,x) \rangle > 0,$$

By quasiinvexity of f, we have $f(x) < f(y)$. Hence again quasiinvexity of f implies that

$$\langle v, \eta(y,x)\rangle \leqslant 0.$$

Therefore (2) holds.

Suppose (2) holds. Assume that there exist $x, y \in K$ such that $f(y) \leqslant f(x)$ and a $\bar{\lambda} \in (0,1)$ such that

$$f(y+\bar{\lambda}\eta(x,y)) > f(x) \geqslant f(y). \tag{4.6}$$

By the Mean value Theorem, there exist $\lambda_1, \lambda_2 \in (0,1), 0<\lambda_2<\bar{\lambda}<\lambda_1<1$ and $\zeta \in \partial^c f(y+\lambda_1\eta(x,y))$ such that

$$f(y+\bar{\lambda}\eta(x,y)) - f(y+\eta(x,y)) = (\bar{\lambda}-1)\langle\zeta, \eta(x,y)\rangle, \tag{4.7}$$

and there exists $\gamma \in \partial^c f(y+\lambda_2\eta(x,y))$ such that

$$f(y+\bar{\lambda}\eta(x,y)) - f(y)) = \bar{\lambda}\langle\gamma, \eta(x,y)\rangle. \tag{4.8}$$

From Assumption C and Remark 3.2, we have

$$\begin{aligned}
&\eta(y+\lambda_2\eta(x,y), y+\lambda_1\eta(x,y)) \\
&\quad = \eta(y+\lambda_2\eta(x,y), y+\lambda_2\eta(x,y)+(\lambda_1-\lambda_2)\eta(x,y)) \\
&\quad = \eta\left(y+\lambda_2\eta(x,y), y+\lambda_2\eta(x,y)+\frac{\lambda_1-\lambda_2}{1-\lambda_2}\eta(x, y+\lambda_2\eta(x,y))\right) \\
&\quad = \frac{\lambda_2-\lambda_1}{1-\lambda_2}\eta(x, y+\lambda_2\eta(x,y)) = (\lambda_2-\lambda_1)\eta(x,y),
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
&\eta(y+\lambda_1\eta(x,y), y+\lambda_2\eta(x,y)) \\
&\quad = \eta(y+\lambda_2\eta(x,y)+(\lambda_1-\lambda_2)\eta(x,y), y+\lambda_2\eta(x,y)) \\
&\quad = \eta\left(y+\lambda_2\eta(x,y)+\frac{\lambda_1-\lambda_2}{1-\lambda_2}\eta(x, y+\lambda_2\eta(x,y)), y+\lambda_2\eta(x,y)\right) \\
&\quad = \frac{\lambda_1-\lambda_2}{1-\lambda_2}\eta(x, y+\lambda_2\eta(x,y)) = (\lambda_1-\lambda_2)\eta(x,y).
\end{aligned} \tag{4.10}$$

Then, from (4.6), (4.7), (4.9) and inequality (B), we obtain

$$\begin{aligned}
&0 \leqslant f(x) - f(y+\eta(x,y)) < f(y+\bar{\lambda}\eta(x,y)) - f(y+\eta(x,y)) \\
&\quad = (1-\bar{\lambda})\langle\zeta, -\eta(x,y)\rangle = \frac{1-\bar{\lambda}}{\lambda_1-\lambda_2}\langle\zeta, \eta(y+\lambda_2\eta(x,y), y+\lambda_1\eta(x,y))\rangle.
\end{aligned}$$

Hence, we have

$$0 < \langle \zeta, \eta(y+\lambda_2\eta(x, y), y+\lambda_1\eta(x, y)) \rangle. \tag{4.11}$$

From (4.6), (4.8) and (4.10), we deduce that

$$\begin{aligned} 0 &< f(y+\bar{\lambda}\eta(x, y)) - f(y) \\ &= \bar{\lambda}\langle \gamma, \eta(x, y)\rangle = \frac{\bar{\lambda}}{\lambda_1-\lambda_2}\langle \gamma, \eta(y+\lambda_1\eta(x, y), y+\lambda_2\eta(x, y))\rangle. \end{aligned}$$

Thus, we have

$$0 < \langle \gamma, \eta(y+\lambda_1\eta(x, y), y+\lambda_2\eta(x, y))\rangle. \tag{4.12}$$

Two inequalities (4.11) and (4.12) contradict the invariant quasimonotonicity of $\partial^c f$ with respect to η. In the case where $f(x) < f(y) < f(y+\overline{\lambda}\eta(x, y))$, the proof is similar. □

EXAMPLE 4.1. Let $K=\mathbb{R}$, $\eta(x, y)=3(x-y)$, and $f(x)=|x|$. Then f is quasiinvex with respect to η but not prequasiinvex with respect to η.

Now we present a non-differentiable version of Theorem 3.1 of Peng (2005).

THEOREM 4.2. *Let M be an open convex subset of X and $f: M \to \mathbb{R}$ locally Lipschitz on M. Suppose that η is affine in the first argument and for each $x \in M$, $\eta(x, x)=0$, and for each $x \neq y$,*

$$f(x) \geqslant f(y) \Rightarrow \exists \bar{x} \in (x, y), \quad \exists u \in \partial^c f(\bar{x}), \quad \langle u, \eta(x, \bar{x})\rangle > 0.$$

If $\partial^c f$ is invariant quasimonotone with respect to η on M, then f is quasiinvex with respect to η on M.

Proof. Let $x, y \in M$, such that

$$f(y) \leqslant f(x). \tag{4.13}$$

If $x=y$, then for each $\zeta \in \partial^c f(x)$, we have

$$\langle \zeta, \eta(y, x)\rangle = 0.$$

If $x \neq y$ by (4.13) and our conditions there exist $\bar{\lambda} \in (0, 1)$ and $u \in \partial^c f(\bar{x})$ such that $\langle u, \eta(x, \bar{x})\rangle > 0$, where $\bar{x}=\bar{\lambda}x+(1-\bar{\lambda})y$. Invariant quasimonotonocity of $\partial^c f$ implies that for each $\zeta \in \partial^c f(x)$, we have

$$\langle \zeta, \eta(\bar{x}, x)\rangle \leqslant 0. \tag{4.14}$$

By the hypothesis on η, we deduce

$$\bar{\lambda}\langle\zeta, \eta(x,x)\rangle + (1-\bar{\lambda})\langle\zeta, \eta(y,x)\rangle = \langle\zeta, \eta(\bar{x}, x)\rangle \leqslant 0.$$

Since $0 < \bar{\lambda} < 1$ and $\eta(x,x) = 0$, we have

$$\langle\zeta, \eta(y,x)\rangle \leqslant 0.$$

Hence, f is a quasiinvex function with respect to η on M. □

THEOREM 4.3. *Let f be locally Lipschitz on K, η satisfy Assumption C and for each $x \neq y$ there exist $\bar{\lambda} \in (0,1)$ and $\zeta \in \partial^c f(y + \bar{\lambda}\eta(x,y))$ such that $\langle\zeta, \eta(x,y)\rangle < 0$. If $\partial^c f$ is invariant quasimonotone with respect to η on K, then f is quasiinvex with respect to η on K.*

Proof. Suppose that $\partial^c f$ is invariant quasimonotone with respect to η on K. Let $x, y \in K$ and $f(y) \geqslant f(x)$. If $x = y$, by Assumption C, $\eta(x,y) = 0$. Let $x \neq y$, then there exist $\bar{\lambda} \in (0,1)$ and $\zeta \in \partial^c f(y + \bar{\lambda}\eta(x,y))$ such that $\langle\zeta, \eta(x,y)\rangle < 0$. From Assumption C, we obtain

$$\langle\zeta, \eta(y, y + \bar{\lambda}\eta(x,y))\rangle > 0.$$

Invariant quasimonotonicity of $\partial^c f$ implies that for each $\gamma \in \partial^c f(y)$,

$$\langle\gamma, \eta(y + \bar{\lambda}\eta(x,y), y)\rangle \leqslant 0.$$

Then by Remark 3.2, for each $\gamma \in \partial^c f(y)$, we have

$$\langle\gamma, \eta(x,y)\rangle \leqslant 0.$$

Hence f is quasiinvex with respect to η on K. □

REMARK 4.1. Theorem 4.3 improves Theorem 3.1 of Yang et al. (2005).

5. Pseudoinvexity and Invariant Pseudomonotonicity

In this section, we establish the relationships between pseudoinvexity of the non-differentiable function f and pseudomonotonicity of the set-valued mapping $\partial^c f$.

DEFINITION 5.1. Let f be locally Lipschitz on K. Then:

(1) function f is said to be pseudoinvex (PIX) with respect to η on K if for any $x, y \in K$ and any $\zeta \in \partial^c f(x)$, one has

$$\langle\zeta, \eta(y,x)\rangle \geqslant 0 \Rightarrow f(y) \geqslant f(x);$$

(2) function f is said to be strictly pseudoinvex (SPIX) with respect to η on K if for any $x, y \in K$ with $x \neq y$ and any $\zeta \in \partial^c f(x)$, one has

$$\langle \zeta, \eta(y, x) \rangle \geqslant 0 \Rightarrow f(y) > f(x);$$

(3) f is said to be strongly pseudoinvex (SGPIX) with respect to η on K if there exists a constant $\alpha > 0$ such that for any $x, y \in K$ and any $\zeta \in \partial^c f(x)$, one has

$$\langle \zeta, \eta(y, x) \rangle \geqslant 0 \Rightarrow f(y) \geqslant f(x) + \alpha \|\eta(y, x)\|^2.$$

By Definitions 3.4, 4.3 and 5.1, we have (SIX)$\Rightarrow$(SPIX), and

$$\begin{array}{ccccc} \text{(SGIX)} & \Longrightarrow & \text{(IX)} & \Longrightarrow & \text{(QIX)} \\ \Downarrow & & \Downarrow & & \\ \text{(SGPIX)} & \Longrightarrow & \text{(PIX)} & & \end{array}$$

DEFINITION 5.2. Let $T: X \to 2^{X^*}$ be a set-valued mapping:

(1) T is said to be an invariant pseudomonotone (IPM) operator on K with respect to η if, for any $x, y \in K$ and any $u \in T(x), v \in T(y)$, one has

$$\langle u, \eta(y, x) \rangle \geqslant 0 \Rightarrow \langle v, \eta(x, y) \rangle \leqslant 0;$$

(2) T is said to be a strictly invariant pseudomonotone (SIPM) operator on K with respect to η if, for any $x, y \in K$ with $x \neq y$ and any $u \in T(x), v \in T(y)$, one has

$$\langle u, \eta(y, x) \rangle \geqslant 0 \Rightarrow \langle v, \eta(x, y) \rangle < 0;$$

(3) T is said to be a strongly invariant pseudomonotone (SGIPM) operator on K with respect to η if there exists a constant $\alpha > 0$ such that for any $x, y \in K$ and any $u \in T(x), v \in T(y)$, one has

$$\langle u, \eta(y, x) \rangle \geqslant 0 \Rightarrow \langle v, \eta(x, y) \rangle \leqslant -\alpha \|\eta(y, x)\|^2.$$

By the Definitions 3.2, 4.1 and 5.2, we have

$$\begin{array}{ccccc} \text{(SGIM)} & \Longrightarrow & \text{(IM)} & & \\ \Downarrow & & \Downarrow & & \\ \text{(SGIPM)} & \Longrightarrow & \text{(IPM)} & \Longrightarrow & \text{(IQM)} \end{array}$$

THEOREM 5.1. *Let f be locally Lipschitz on K and, f and η satisfy Assumptions A and C, respectively. If $\partial^c f$ is invariant pseudomonotone with respect to η on K, then f is pseudoinvex with respect to η on K.*

Proof. Suppose that $\partial^c f$ is invariant pseudomonotone with respect to η on K. Let $x, y \in K$, and

$$f(y) < f(x). \tag{5.1}$$

By the Mean Valued Theorem, there exist $\bar{\lambda} \in (0,1)$ and $\gamma \in \partial^c f(x + \bar{\lambda}\eta(y,x))$ such that

$$f(x+\eta(y,x)) - f(x) = \langle \gamma, \eta(y,x) \rangle. \tag{5.2}$$

By Assumptions A and C, it follows that

$$f(x+\eta(y,x)) \leqslant f(y), \tag{5.3}$$

and

$$\eta(x, x+\bar{\lambda}\eta(y,x)) = -\bar{\lambda}\eta(y,x). \tag{5.4}$$

Now, from (5.1) to (5.4), we have

$$\langle \gamma, \eta(x, x+\bar{\lambda}\eta(y,x)) \rangle > 0. \tag{5.5}$$

Since $\partial^c f$ is invariant pseudomonotone with respect to η, it follows from (5.5) for each $\zeta \in \partial^c f(x)$ we have

$$\langle \zeta, \eta(x+\bar{\lambda}\eta(y,x), x) \rangle < 0.$$

From Remark 3.2, we obtain

$$\langle \zeta, \eta(y,x) \rangle < 0.$$

Hence, f is pseudoinvex with respect to η. □

REMARK 5.1. Theorem 5.1 improves Lemma 4.1 of Yang et al. (2003).

LEMMA 5.1. *Let f be locally Lipschitz on K. If f is strictly pseudoinvex with respect to η on K, then $\partial^c f$ is strictly invariant pseudomonotone with respect to η on K.*

Proof. Suppose that f is strictly pseudoinvex with respect to η on K. Let $x, y \in K$, $x \neq y$, and $\zeta \in \partial^c f(x)$ be such that

$$\langle \zeta, \eta(y,x) \rangle \geqslant 0. \tag{5.6}$$

We need to show that for any $\gamma \in \partial^c f(y)$ we have $\langle \gamma, \eta(x, y)\rangle < 0$. On the contrary, we assume that

$$\langle \gamma, \eta(x, y)\rangle \geqslant 0.$$

From the strict pseudoinvexity of f with respect to η, it follows that

$$f(x) > f(y). \tag{5.7}$$

On the other hand, the strict pseudoinvexity of f with respect to η and (5.6) imply that

$$f(y) > f(x),$$

which contradicts (5.7). □

THEOREM 5.2. *Let f be locally Lipschitz on K and, f and η satisfy Assumptions A and C, respectively. If $\partial^c f$ is strictly invariant pseudomonotone with respect to η on K, then f is strictly pseudoinvex with respect to η on K.*

Proof. Suppose that $\partial^c f$ is strictly invariant pseudomonotone with respect to η on K. Let $x, y \in K$, $x \neq y$, and

$$f(y) \leqslant f(x). \tag{5.8}$$

By the Mean Value Theorem, there exist $\bar{\lambda} \in (0, 1)$ and $\gamma \in \partial^c f(x + \bar{\lambda}\eta(y, x))$ such that

$$f(x + \eta(y, x)) - f(x) = \langle \gamma, \eta(y, x)\rangle. \tag{5.9}$$

By Assumptions A,

$$f(x + \eta(y, x)) \leqslant f(y). \tag{5.10}$$

Now, from (5.8) to (5.10) and Assumption C, we have

$$\langle \gamma, \eta(x, x + \bar{\lambda}\eta(y, x))\rangle = -\bar{\lambda}\langle \gamma, \eta(y, x)\rangle \geqslant 0. \tag{5.11}$$

Since $\partial^c f$ is strictly invariant pseudomonotone with respect to η, we conclude that for each $\zeta \in \partial^c f(x)$,

$$\langle \zeta, \eta(x + \bar{\lambda}\eta(y, x), x)\rangle < 0. \tag{5.12}$$

From Remark 3.2, we deduce

$$\eta(x+\bar{\lambda}\eta(y,x),x)=\bar{\lambda}\eta(y,x).$$

Thus, it follows from (5.12) that

$$\langle\zeta,\eta(y,x)\rangle<0.$$

Therefore, f is strictly pseudoinvex with respect to η on K. □

REMARK 5.2. Lemma 5.1 and Theorem 5.2 improve Theorem 5.1 of Yang et al. (2003).

THEOREM 5.3. *Let f be locally Lipschitz on K and, f and η satisfy Assumptions A and C, respectively. If $\partial^c f$ is strongly invariant pseudomonotone with respect to η on K, then f is strongly pseudoinvex with respect to η on K.*

Proof. Suppose that $\partial^c f$ is strongly invariant pseudomonotone with respect to η on K. Let $x, y \in K$, and $\zeta \in \partial^c f(x)$ be such that

$$\langle\zeta,\eta(y,x)\rangle\geqslant 0.$$

Set $z=x+1/2\eta(y,x)$. By the Mean Value Theorem, there exist $\lambda_1,\lambda_2\in(0,1), 0<\lambda_2<1/2<\lambda_1<1$, $\sigma\in\partial^c f(u)$, where $u=x+\lambda_2\eta(y,x)$ such that

$$f(z)-f(x)=\langle\sigma,z-x\rangle=\frac{1}{2}\langle\sigma,\eta(y,x)\rangle.$$

Hence by Assumption C, we have

$$f(z)-f(x)=\frac{-1}{2\lambda_2}\langle\sigma,\eta(x,u)\rangle, \tag{5.13}$$

and there exists $\gamma\in\partial^c f(v)$ where $v=x+\lambda_1\eta(y,x)$, such that

$$f(x+\eta(y,x))-f(z)=\langle\gamma,x+\eta(y,x)-z\rangle=\frac{1}{2}\langle\gamma,\eta(y,x)\rangle.$$

Thus by Assumption C, we have

$$f(x+\eta(y,x))-f(z)=-\frac{1}{2\lambda_1}\langle\gamma,\eta(x,v)\rangle. \tag{5.14}$$

On the other hand by Remark 3.2, we derive

$$0\leqslant\langle\zeta,\eta(y,x)\rangle=\frac{1}{\lambda_1}\langle\zeta,\eta(v,x)\rangle=\frac{1}{\lambda_2}\langle\zeta,\eta(u,x)\rangle.$$

By strongly invariant pseudomonotonicity of $\partial^c f$, we obtain

$$\langle \sigma, \eta(x, u) \rangle \leqslant -\alpha \|\eta(u, x)\|^2 = -\alpha \lambda_2^2 \|\eta(y, x)\|^2,$$

and

$$\langle \gamma, \eta(x, v) \rangle \leqslant -\alpha \|\eta(v, x)\|^2 = -\alpha \lambda_1^2 \|\eta(y, x)\|^2.$$

Then by (5.13)

$$f(z) - f(x) \geqslant \frac{\alpha}{2} \lambda_2 \|\eta(y, x)\|^2,$$

and by (5.14), we deduce that

$$f(x + \eta(y, x)) - f(z) \geqslant \frac{\alpha}{2} \lambda_1 \|\eta(y, x)\|^2.$$

Adding these two inequalities, we have

$$f(x + \eta(y, x)) - f(x) \geqslant \frac{\alpha}{2} (\lambda_1 + \lambda_2) \|\eta(y, x)\|^2,$$

therefore by Assumption A, we obtain

$$f(y) - f(x) \geqslant \frac{\alpha}{4} \|\eta(y, x)\|^2.$$ □

REMARK 5.3. Theorem 5.3 improves Theorem 4.3 of Fan et al. (2003).

THEOREM 5.4. *Let M be an open convex subset of X and $f: M \to \mathbb{R}$ be locally Lipschitz on M. Suppose that η is affine in the first argument and for each $x \in M$, $\eta(x, x) = 0$, and for each $x, y \in M$*

$$f(x) > f(y) \Rightarrow \exists \bar{x} \in (x, y), \quad \exists u \in \partial^c f(\bar{x}), \quad \langle u, \eta(x, \bar{x}) \rangle > 0.$$

If $\partial^c f$ be invariant pseudomonotone with respect to η on M, then f is pseudoinvex with respect to η on M.

Proof. Suppose that $\partial^c f$ is invariant pseudomonotone with respect to η on M. Let $x, y \in M$, and

$$f(y) < f(x). \tag{5.15}$$

By our conditions, there exist $\bar{\lambda} \in (0, 1)$ and $u \in \partial^c f(\bar{x})$ such that $\langle u, \eta(x, \bar{x}) \rangle > 0$, where $\bar{x} = \bar{\lambda} x + (1 - \bar{\lambda}) y$. Invariant pseudomonotonicity of $\partial^c f$ implies that for each $\zeta \in \partial^c f(x)$, we have

$$\langle \zeta, \eta(\bar{x}, x) \rangle < 0. \tag{5.16}$$

By the hypothesis on η, we have

$$\bar{\lambda}\langle\zeta, \eta(x, x)\rangle + (1-\bar{\lambda})\langle\zeta, \eta(y, x)\rangle = \langle\zeta, \eta(\bar{x}, x)\rangle < 0.$$

Since $0 < \bar{\lambda} < 1$ and $\eta(x, x) = 0$, we have

$$\langle\zeta, \eta(y, x)\rangle < 0.$$

Then f is pseudoinvex with respect to η on M. □

REMARK 5.4. Theorem 5.4 improves Theorem 2.2 of Peng (2005).

In the following theorem we obtain an analogous result to Theorem 2.1 of Peng (2005) in our context.

THEOREM 5.5. *Let M be an open convex subset of X and $f: M \to \mathbb{R}$ be locally Lipschitz on M. Suppose that η is affine in the first argument and for each $x \in M, \eta(x, x) = 0$, and for each $x \neq y$,*

$$f(x) \geqslant f(y) \Rightarrow \exists \bar{x} \in (x, y), \quad \exists u \in \partial^c f(\bar{x}), \quad \langle u, \eta(x, \bar{x})\rangle \geqslant 0.$$

If $\partial^c f$ be strictly invariant pseudomonotone with respect to η on M, then f is strictly pseudoinvex with respect to η on M.

Proof. Suppose that $\partial^c f$ is strictly invariant pseudomonotone with respect to η on M. Let $x, y \in M, x \neq y$ and

$$f(y) \leqslant f(x). \tag{5.17}$$

By our conditions, there exist $\bar{\lambda} \in (0, 1)$ and $u \in \partial^c f(\bar{x})$ such that $\langle u, \eta(x, \bar{x})\rangle \geqslant 0$, where $\bar{x} = \bar{\lambda}x + (1-\bar{\lambda})y$. Strictly invariant pseudo monotonicity of $\partial^c f$ implies that for each $\zeta \in \partial^c f(x)$, we have

$$\langle\zeta, \eta(\bar{x}, x)\rangle < 0. \tag{5.18}$$

By our hypothesis on η, we obtain

$$\bar{\lambda}\langle\zeta, \eta(x, x)\rangle + (1-\bar{\lambda})\langle\zeta, \eta(y, x)\rangle = \langle\zeta, \eta(\bar{x}, x)\rangle < 0.$$

Since $0 < \bar{\lambda} < 1$ and $\eta(x, x) = 0$, we have

$$\langle\zeta, \eta(y, x)\rangle < 0.$$

Thus, f is strictly pseudoinvex with respect to η on M. □

6. Applications

In this section we establish some applications of our results for the solution of the variational like inequality problems as well as the mathematical programming problems. We give first some definitions.

DEFINITION 6.1. Let M be subset of a topological space X. A set-valued mapping. $T: X \to 2^{X^*}$ is said to be upper semi-continuous, if for each weak*-topology closed set $B \subset X^*$, $T^-(B) = \{x \in X: T(x) \cap B \neq \emptyset\}$ is closed in X. T is called upper hemicontinuous if its restriction to line segments of its domain is upper semicontinuous where E^* is equipped with the weak*-topology.

The following Lemmas can be viewed as extensions and generalizations of Minty's Lemma; see Lemma 2.3 of Noor (2005).

LEMMA 6.1. *Let K be an invex set with respect to η and $T: K \to 2^{X^*}$ be an invariant pseudomonotone map with respect to η. Suppose that*

(1) *T is upper hemicontinuous,*
(2) *η satisfies Assumption C.*

Then for $x \in K$, the following assertions are equivalent:

(a) *for each $y \in K$, there exists $u \in T(x)$, such that $\langle u, \eta(y, x)\rangle \geqslant 0$.*
(b) *for each $y \in K$ and for each $v \in T(y)$, we have $\langle v, \eta(x, y)\rangle \leqslant 0$.*

Proof. $(a) \Rightarrow (b)$ deduces from the definition of invariant pseudomonotone with respect to η. Conversely, let $x \in K$ be a solution of (b) and $y \in K$, we consider $w_t = x + t\eta(y, x)$, with $0 < t < 1$, Hence by (b), for each $v_t \in T(w_t)$, we have

$$\langle v_t, \eta(x, w_t)\rangle \leqslant 0.$$

From Assumption C, we have

$$\eta(x, w_t) = \eta(x, x + t\eta(y, x)) = -t\eta(y, x),$$

then

$$\langle v_t, \eta(y, x)\rangle \geqslant 0 \quad \text{for each } t \in (0, 1) \quad \text{and each } v_t \in T(w_t).$$

Suppose in the contrary, $\langle u, \eta(y, x)\rangle < 0$ for all $u \in T(x)$. Then by upper hemicontinuity of T, we have $\langle u, \eta(y, x)\rangle < 0$ for all $u \in T(w_t)$ and for sufficiently small t, which is a contradiction. □

DEFINITION 6.2. The function $\eta: X \times X \to X$ is called a skew function, if $\eta(x, y) + \eta(y, x) = 0$ for each $x, y \in X$.

LEMMA 6.2. *Let K be a convex subset of X and $T: K \to 2^{X^*}$ be an invariant pseudomonotone operator with respect to η. Suppose that*

(1) *T is upper hemicontinuous,*
(2) *η is affine in the first argument and skew function.*

Then for $x \in K$, the following assertions are equivalent

(a) *for each $y \in K$, there exists $u \in T(x)$ such that $\langle u, \eta(y, x)\rangle \geqslant 0$.*
(b) *for each $y \in K$ and for each $v \in T(y)$, we have $\langle v, \eta(x, y)\rangle \leqslant 0$.*

Proof. $(a) \Rightarrow (b)$ deduces from the definition of invariant pseudomonotonicity of T with respect to η on K. Conversely, let $x \in K$ be a solution of (b) and $y \in K$. We consider $w_t = ty + (1-t)x$, with $0 < t < 1$. Hence by (b), for each $v_t \in T(w_t)$, we have

$$\langle v_t, \eta(x, w_t)\rangle \leqslant 0.$$

Since η is skew,

$$\langle v_t, \eta(w_t, x)\rangle \geqslant 0.$$

As η is affine in the first argument and η is skew, hence $\eta(u, u) = 0$, for each $u \in K$ and therefore,

$$\langle v_t, \eta(y, x)\rangle \geqslant 0.$$

Suppose in the contrary, $\langle u, \eta(y, x)\rangle < 0$ for all $u \in T(x)$. Then by upper hemicontinuity of T, we have $\langle u, \eta(y, x)\rangle < 0$ for all $u \in T(w_t)$ and for sufficiently small t, which is a contradiction. □

The mathematical programming problem (MP) is defined as

$$\text{(MP)} \quad \min f(x) \quad s.t. \ x \in K,$$

where $f: K \subseteq X \to \mathbb{R}$.

THEOREM 6.1. *Let $f: K \to \mathbb{R}$ be locally Lipschitz on K where K is an invex set. For $x \in K$, set*

(1) *for each $y \in K$, there exists $u \in \partial^c f(x)$ such that $\langle u, \eta(y, x)\rangle \geqslant 0$,*
(2) *x is a solution of the (MP) Problem,*
(3) *for each $y \in K$ and for each $v \in \partial^c f(y)$ we have $\langle v, \eta(x, y)\rangle \leqslant 0$.*

Then we have;

(a) *If f is pseudoinvex with respect to η on K, then* (1) $\Rightarrow$ (2).
(b) *If f is quasiinvex with respect to η on K, then* (2) $\Rightarrow$ (3).
(c) *If f is invex with respect to η on K and either η satisfies Assumption C or η is affine in the first argument, skew function and K is also convex, then* (1), (2) *and* (3) *are equivalent.*
(d) *If f is strictly pseudoinvex with respect to η on K and either η satisfies Assumption C or η is affine in the first argument, skew function and K is also convex, then* (1), (2) *and* (3) *are equivalent and in this case x is unique.*

Proof.

(a) Follows from Definition 5.1.
(b) Follows from Definition 4.3.
(c) If f is invex with respect to η on K, then f is pseudoinvex with respect to η on K and by (a), (1) $\Rightarrow$ (2). Since f is quasiinvex with respect to η on K then by (b), (2) $\Rightarrow$ (3), and by Lemma 3.1 $\partial^c f$ is invariant monotone and hence it is invariant pseudomonotone with respect to η on K. Now by Lemmas 2.1, 6.1 and 6.2, we have (3) $\Rightarrow$ (1).
(d) If f is strictly pseudoinvex with respect to η on K, then f is pseudoinvex with respect to η on K and hence by (a), (1) $\Rightarrow$ (2). By Definitions 4.3, 5.1, we have f is quasiinvex with respect to η on K, therefore by (b), (2) $\Rightarrow$ (3). Lemma 5.1 implies $\partial^c f$ is a strictly invariant pseudomonotone operator and hence it is an invariant pseudomonotone operator with respect to η on K, then by Lemmas 2.1, 6.1 and 6.2, we have (3) $\Rightarrow$ (1). The uniqueness of x follows from Definition 5.1. □

Now we will establish a solution for the variational-like problems.

DEFINITION 6.3. A set valued map $\Gamma : M \to 2^M$ is called KKM-map if for every finite subset $\{x_1, x_2, \dots, x_n\}$ of M,

$$\text{conv}(\{x_1, x_2, \dots, x_n\}) \subset \bigcup_{i=1}^{n} \Gamma(x_i).$$

In the following we obtain a refinement of Theorem 5.1 of Garzon et al. (2003) in non-compact setting.

THEOREM 6.2. *Let M be a non-empty closed convex subset of X and $T : M \to 2^{X^*}$ be an invariant pseudomonotone map with respect to η. Suppose that:*

(1) *T is upper hemicontinous,*
(2) *η is affine and continuous in the first argument and skew function,*
(3) *there is a non-empty compact set $K \subseteq M$, and there is a non-empty compact convex set $B \subseteq M$ such that for each $x \in M \backslash K$, there exist $y \in B$ and $u \in T(y)$ such that $\langle u, \eta(x, y)\rangle > 0$.*

Then there is $x_0 \in M$, such that for each $y \in M$, there exists $u \in T(x_0)$ such that $\langle u, \eta(y, x_0)\rangle \geqslant 0$.

Proof. Define the set-valued map $\hat{\Gamma} : M \to 2^M$, as

$$\hat{\Gamma}(y) = \{x \in M : \exists u \in T(x) \quad s.t.\ \langle u, \eta(y, x)\rangle \geqslant 0\}, \quad \text{for each } y \in M.$$

First, we prove that $\hat{\Gamma}(x)$ is a KKM-map.

Suppose that $\{y_1, y_2, \dots, y_n\} \subset M$, $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i \geqslant 0$, $i = 1, 2, \dots, n$, and

$$y = \sum_{i=1}^n \alpha_i y_i \notin \bigcup_{i=1}^n \hat{\Gamma}(y_i).$$

Then we have

$$\forall i = 1, 2, \dots, n\ \forall v \in T(y), \quad \langle v, \eta(y_i, y)\rangle < 0.$$

Since η is affine in the first argument,

$$\forall v \in T(y), \quad \left\langle v, \eta\left(\sum_{i=1}^n \alpha_i y_i, y\right)\right\rangle = \sum_{i=1}^n \alpha_i \langle v, \eta(y_i, y)\rangle < 0,$$

then

$$\forall v \in T(y), \quad \langle v, \eta(y, y)\rangle < 0,$$

which contradicts the assumption of skewness, which demands that

$$\eta(y, y) = 0, \quad \forall y \in M.$$

So we derive

$$\operatorname{conv}(\{y_1, y_2, \dots, y_n\}) \subset \bigcup_{i=1}^n \hat{\Gamma}(y_i)$$

and therefore, $\hat{\Gamma}(x)$ is a KKM-map.

Define the set-valued map $\Gamma : M \to 2^M$, such that

$$\Gamma(y) = \{x \in M : \forall v \in T(y), \langle v, \eta(x, y)\rangle \leqslant 0\} \quad \text{for all } y \in M.$$

Since T is (IPM), then $\hat{\Gamma}(y) \subset \Gamma(y)$ for all $y \in M$ and since $\hat{\Gamma}$ is a KKM-map, then Γ is also a KKM-map.

On the other hand by Lemma 6.2,

$$\bigcap_{y\in M} \hat{\Gamma}(y) = \bigcap_{y\in M} \Gamma(y).$$

Moreover, $\Gamma(y)$ for every $y \in M$ is closed, since η is continuous in the first argument. Condition (3) implies that $\bigcap_{y\in B} \Gamma(y) \subseteq K$. Therefore, Theorem 2.1 of Fakhar and Zafarani (2005) implies that

$$\bigcap_{y\in M} \hat{\Gamma}(y) = \bigcap_{y\in M} \Gamma(y) \neq \emptyset.$$

Hence, there exists $x_0 \in M$ such that for each $y \in M$ there exists $u \in T(x_0)$ such that $\langle u, \eta(y, x_0)\rangle \geqslant 0$. □

REMARK 6.2. Condition (3) can be replaced by the following condition: (3′) there exist a non-empty compact subset K of M and a finite subset A of M such that for every $x \in M \backslash K$, there exist $y \in A$ and $u \in T(y)$ such that $\langle u, \eta(x, y)\rangle > 0$. In fact in this case we can use Lemma 2.1 of Fakhar and Zafarani (2004) instead of Theorem 2.1 of Fakhar and Zafarani (2005).

In the next result we will establish the uniqueness of the above solution.

COROLLARY 6.1. *Let M be a non-empty convex subset of X and $T: M \to 2^{X^*}$ be a strictly variant pseudomonotone operator with respect to η. Suppose that conditions* (1–3) *of Theorem* 6.2 *hold. Then there is a unique $x_0 \in M$ such that for each $y \in M$ there exists $u \in T(x_0)$ such that $\langle u, \eta(y, x_0)\rangle \geqslant 0$.*

Proof. As (SIPM) $\Rightarrow$ (IPM) and by Theorem 6.2 we have the existence of a solution x_0. Now we prove the uniqueness.

Suppose that we have two distinct solutions, $x_0, x_1 \in M$. Then there exists $u \in T(x_0)$ such that

$$\langle u, \eta(x_1, x_0)\rangle \geqslant 0, \tag{6.1}$$

and there exists $v \in T(x_1)$ such that

$$\langle v, \eta(x_0, x_1)\rangle \geqslant 0. \tag{6.2}$$

Since T is (SIPM), it follows from (6.1) that $\langle v, \eta(x_0, x_1)\rangle < 0$, which contradicts (6.2). □

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